Nonlinear Vibration of Cylindrical Shells

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The large amplitude vibration of a thin-walled cylindrical shell is analyzed. A perturbation method is used to solve the steady-state forced vibration problem. The simply-supported boundary conditions and the circumferential periodicity condition are satisfied. The resulting solution indicates that in addition to the fundamental mode, the response contains asymmetric modes as well as axisymmetric modes with the frequency twice that of the fundamental mode. Vibrations involving a single driven mode response are investigated. The results indicate that the nonlinearity is either softening or hardening depending on the mode. The vibrations involving both a driven mode and a companion mode are also investigated. An experimental investigation is conducted, and the results are in qualitative agreement with the theory. "Nonstationary" response is detected at some frequencies for large amplitude response in which the amplitude drifts from one value to another. Various nonlinear phenomena are observed and comparisons with the theoretical results are made.

I. Introduction

THE nonlinear effects of large amplitude vibration of cylindrical shells are demonstrated by two phenomena; namely, the response-frequency relationship in the vicinity of a resonant frequency and the occurrence of traveling wave response. Contrasted to the linear vibration in which the resonant frequency is independent of its amplitude of vibration, the resonant frequency in nonlinear vibration is a function of its amplitude. The response-frequency relationship will indicate whether the nonlinearity is of hardening type (frequency increasing with amplitude) or softening type (frequency decreasing with amplitude). Similar phenomena can be observed on a much simpler system such as a spring-mass system with a nonlinear characteristic in the spring.[1]

In the linear vibration of cylindrical shells, the response to a stationary periodic excitation is in the form of a standing wave. This standing wave will be in conformity with the distribution of the external excitation. In case of the external force applied at a discrete point, the standing wave will be symmetrical with respect to the point of application. In the present investigation this standing wave will be referred to as a "driven mode."

A striking feature of the results of experimental observation of supersonic cylindrical shell flutter[2] is that "almost all the flutter modes observed in these experiments were of the circumferentially traveling-wave type." Such traveling waves are not predicted by linear theory, and it was suggested in Ref. 2 that nonlinearities in the shell were responsible for the phenomenon. A circumferentially traveling wave in cylindrical shell response can be decomposed into a driven mode standing wave and another standing wave which is circumferentially 90° out of phase with the driven mode (which will be referred to as a "companion mode" in this investigation). Therefore, the occurrence of the circumferentially traveling wave is a result of the companion mode being excited due to the nonlinearities in the shell response. Related studies on nonlinear vibrations of circular rings[3] and circular plates[4] have shown the need for including these "companion modes" in the study of nonlinear forced vibrations of axisymmetric structures. In addition to the supersonic cylindrical shell flutter, the traveling wave responses have been observed in the sloshing-vibration of thin cylindrical shells.[5]

The problem of large amplitude vibration of cylindrical shells has given rise to a number of theoretical studies.[6-16] The basic approach used in these studies (except Ref. 16) is to assume the shape of the deflection, that is, the shape of the vibration mode, sometimes referred to as generalized coordinates, and then to derive a set of nonlinear ordinary differential equations by using Galerkin's approximation procedure. Subsequently, the nonlinear ordinary differential equations are solved to obtain the response-frequency relationship.

The Galerkin method is a very powerful approximation method that reduces a system of nonlinear partial differential equations into a system of nonlinear ordinary differential equations. Also the Galerkin method provides insight to the nonlinear coupling of various vibration modes during the solution procedure. However, its results are highly dependent on the assumed deflection shape. Completely different results can be obtained by a small difference in the assumed deflection shape as can be seen from the previous investigations on this subject. Evensen[17] and Dowell[12] have pointed out that the axisymmetric modes in the assumed deflection shape play an important role in the outcome of the results. Yet there is no standard for selecting the particular axisymmetric modes for the assumed deflection shape. A detailed summary of previous investigations of nonlinear vibration of cylindrical shells can be found in Ref. 18.

The present investigation will apply a perturbation technique to reduce the governing nonlinear differential equations, and so-called circumferential periodicity condition, into a system of linear equations. Then a sequence of exact solutions will be obtained without the handicap of preselecting a particular deflection for the solution. The solutions obtained in this fashion, as compared to that of the previous investigations, satisfy the governing equations as well as boundary conditions and circumferential periodicity condition asymptotically. The usual response-frequency relationship will be calculated and the occurrence of traveling wave will be studied. Finally the rsults will be compared with an experimental investigation.

II. Governing Equations

The well-known approximation of Donnell's shallow-shell theory for thin-walled circular cylinders as exemplified in Ref. 7 result in 3 equations of equilibrium. By assuming a stress

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function F, these equations can be combined to give

$$D\nabla^4 w + Ch\frac{\partial w}{\partial t} + \rho h\frac{\partial^2 w}{\partial t^2} = q(x,\,y,\,t) + \frac{1}{R}\frac{\partial^2 F}{\partial x^2} \quad R$$
$$+\left[\frac{\partial^2 F}{\partial y^2}\frac{\partial^2 w}{\partial x^2} - 2\frac{\partial^2 F}{\partial x \partial y}\frac{\partial^2 W}{\partial x \partial y} + \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right] \qquad (1)$$

and

$$\frac{1}{Eh}\nabla^4 F = -\frac{1}{R}\frac{\partial^2 w}{\partial x^2} + \left[\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right] \qquad (2)$$

where

$$q(x,\,y,\,t) = f\cos\omega t\cos(ny/R)\sin(m\pi x/L) \qquad (3)$$

The coordinate system and displacements are shown in Fig. 1.

The boundary conditions in this investigation are assumed to be the "classical simply supported" conditions, sometimes referred to as SS1/SS1 conditions. Mathematically, they can be expressed as

$$w = M_x = N_x = v = 0 \qquad \text{at} \quad x = 0,\ L \qquad (4)$$

where M_x is the bending moment in the x-direction, N_x is the inplane resultant force in the x-direction, and v is the circumferential displacement. These can be expressed in terms of radial displacement w and stress function F as follows

$$M_x = -D[(\partial^2 w/\partial x^2) + \nu(\partial^2 w/\partial y^2)] \qquad (5)$$

$$N_x = (\partial^2 F/\partial y^2) \qquad (6)$$

$$\frac{\partial v}{\partial y} = \frac{1}{Eh}\left[\frac{\partial^2 F}{\partial x^2} - \nu\frac{\partial^2 F}{\partial y^2} + \frac{w}{R} - \frac{1}{2}\left[\frac{\partial w}{\partial y}\right]^2\right] \qquad (7)$$

For a complete cylindrical shell such as is considered here, it is apparent that all the physical quantities such as displacements, and force and moment resultants must satisfy the circumferential periodicity condition. This periodicity condition assures that all the physical quantities will be continuous and single-valued whenever the circumferential coordinate changes by 2π. The dependent variables w and F in Eqs. (1) and (2) must be determined such that all physical quantities satisfy this periodicity condition. The condition which results from this requirement is as follows

$$\int_0^{2\pi R}\frac{\partial v}{\partial y}\,dy = 0$$

$$= \int_0^{2\pi R}\frac{1}{Eh}\left[\frac{\partial^2 F}{\partial x^2} - \nu\frac{\partial^2 F}{\partial y^2} + \frac{w}{R} - \frac{1}{2}\left[\frac{\partial w}{\partial y}\right]^2\right]dy \qquad (8)$$

It may be noted in passing that nonlinear static stability analyses have made use of this circumferentially periodicity condition since the 1941 paper by von Karman and Tsien.[19]

It should be noted that the Donnell shallow-shell equations are limited by various approximations made in their derivation. These approximations limit the validity of the analysis to a certain region. A more detailed discussion and error estimation based on Koiter's shell equations may be found in Ref. 20.

Before proceeding to apply the perturbation method, the differential equations, boundary conditions, and circumferential periodicity condition must be rewritten in the nondimensional form. In the following, the nondimensional variables and nondimensional quantities will be defined.

$$s = \frac{x}{L},\quad \theta = \frac{y}{R},\quad \tau = \omega t$$

$$W(s,\theta,\tau) = \frac{w(x,y,t)}{w_m},\quad V(s,\theta,\tau) = \frac{v(x,y,t)}{w_m}$$

$$\phi(s,\theta,\tau) = \frac{(1-\nu^2)}{EhRw_m}F(x,y,t),\quad k = \frac{L}{R}$$

$$r = \frac{1}{(12)^{1/2}}\frac{h}{R},\quad \gamma = \frac{C}{2\rho\omega_0},\quad \epsilon = \frac{w_m}{R}$$

$$\lambda = \frac{(1-\nu^2)\rho R^2\omega^2}{E},\quad \bar{f} = \frac{(1-\nu^2)R^2}{Ehw_m}f$$

$$\lambda_0 = \frac{(1-\nu^2)\rho R^2\omega_0^2}{E} \qquad (9)$$

where w_m is the maximum radial displacement and ω_0 is the frequency of linear vibration of a cylindrical shell to be defined later.

Upon substitution of Eq. (9) into Eqs. (1, 2, 4, and 8), the following equations may be obtained.

$$r^2\left[\frac{1}{k^4}\frac{\partial^4 W}{\partial s^4} + \frac{2}{k^2}\frac{\partial^4 W}{\partial s^2\partial\theta^2} + \frac{\partial^4 W}{\partial\theta^4}\right]$$
$$+\lambda\frac{\partial^2 W}{\partial\tau^2} + 2\sqrt{\lambda}\sqrt{\lambda_0}\gamma\frac{\partial w}{\partial\tau} = +\bar{f}\cos\tau\cos(n\theta)\sin(m\pi s)$$
$$+\frac{1}{k^2}\frac{\partial^2\phi}{\partial s^2} + \frac{\epsilon}{K^2}\left[\frac{\partial^2\phi}{\partial\theta^2}\frac{\partial^2 W}{\partial s^2} - 2\frac{\partial^2\phi}{\partial\theta\partial s}\frac{\partial^2 W}{\partial\theta\partial s}\right.$$
$$\left.+\frac{\partial^2\phi}{\partial s^2}\frac{\partial^2 W}{\partial\theta^2}\right] \qquad (10a)$$

$$\frac{1}{1-\nu^2}\left[\frac{1}{k^4}\frac{\partial^4\phi}{\partial s^4} + \frac{2}{k^2}\frac{\partial^4\phi}{\partial s^2\partial\theta^2} + \frac{\partial^4\phi}{\partial\theta^4}\right] =$$
$$-\frac{1}{k^2}\frac{\partial^2 W}{\partial s^2} + \frac{\epsilon}{k^2}\left[\left[\frac{\partial^2 W}{\partial\theta\partial s}\right]^2 - \frac{\partial^2 W}{\partial s^2}\frac{\partial^2 W}{\partial\theta^2}\right] \qquad (10b)$$

The boundary condition can be obtained as

$$W = \left[\frac{1}{k^2}\frac{\partial^2 W}{\partial s^2} + \nu\frac{\partial^2 W}{\partial\theta^2}\right]$$
$$= \frac{\partial^2\phi}{\partial\theta^2} = V = 0 \quad \text{at } s = 0, 1 \qquad (11)$$

The circumferential periodicity condition,

$$\int_0^{2\pi}\frac{\partial V}{\partial\theta}d\theta = \int_0^{2\pi}\left[\frac{1}{1-\nu^2}\left[\frac{1}{k^2}\frac{\partial^2\phi}{\partial s^2} - \nu\frac{\partial^2\phi}{\partial\theta^2}\right]\right.$$
$$\left.+\left[W - \frac{1}{2}\epsilon\left[\frac{\partial W}{\partial\theta}\right]^2\right]\right]d\theta = 0 \qquad (12)$$

It is clear now that the nonlinear terms in Eqs. (10-12) are multiplied by a coefficient $\epsilon = (w_m/R)$. In the vibration of cylindrical shells the maximum amplitude of response is of the order of shell thickness h. Since thin-walled shells $(h/R < < 1)$

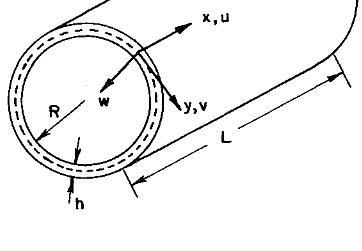


Fig. 1 Shell geometry and coordinate system.

are considered, the coefficient $\epsilon=(w_m/R)$ is a small parameter. The solutions of the governing Eq. (10) and frequency parameter λ will be expanded into the power series of ϵ as follows

$$w(s,\theta,\tau,\epsilon)=\sum_{i=0}^{\infty}\epsilon^i W_i(s,\theta,\tau)$$

$$\phi(s,\theta,\tau,\epsilon)=\sum_{i=0}^{\infty}\epsilon^i \phi_i(s,\theta,\tau)$$

$$V(s,\theta,\tau,\epsilon)=\sum_{i=0}^{\infty}\epsilon^i V_i(s,\theta,\tau)$$

$$\lambda=\sum_{i=0}^{\infty}\epsilon^i\lambda_i \tag{13}$$

Substituting Eq. (13) into Eqs. (10-12) and equating the terms with equal powers of ϵ, a system of linear equations may be obtained. Before this can be carried out, an assumption on the order of the external force $\bar{F}$ and the damping γ, must be made. Since the response of the shell in the vicinity of resonance is desired, the forcing function should be small in some sense since the damping of the shell is small. For the type of shell considered in the test program (seamless with integral rings) the damping is much less than 1%. It is therefore clear that the damping and the forcing function must be at least of order ϵ or higher order to make physical sense. The choice of this order cannot be made from mathematical reasoning alone but must reflect the nature of solution desired from the perturbation procedure. For the problem at hand it appeared reasonable and convenient to assume both the force and damping to be of order ϵ^2. Using this assumption, the linearized system of equations is as follows

order of ϵ^0

$$r^2\left[\frac{1}{k^4}\frac{\partial^4 W_0}{\partial s^4}+\frac{2}{k^2}\frac{\partial^4 W_0}{\partial s^2\partial\theta^2}+\frac{\partial^4 W_0}{\partial\theta^4}\right]+\lambda_0\frac{\partial^2 W_0}{\partial\tau^2}-\frac{1}{k^2}\frac{\partial^2\phi_0}{\partial s^2}=0 \tag{14a}$$

$$\frac{1}{1-\nu^2}\left[\frac{1}{k^4}\frac{\partial^2\phi_0}{\partial s^4}+\frac{2}{k^2}\frac{\partial^4\phi_0}{\partial s^2\partial\theta^2}+\frac{\partial^4\phi_0}{\partial\theta^4}\right]+\frac{1}{k^2}\frac{\partial^2 W_0}{\partial s^2}=0 \tag{14b}$$

boundary condition:

$$W_0=\frac{1}{k^2}\frac{\partial^2 W_0}{\partial s^2}+\nu\frac{\partial^2 W_0}{\partial\theta^2}=\frac{\partial^2\phi_0}{\partial\theta^2}=V_0=0 \quad \text{at } s=0,1 \tag{15}$$

circumferential periodicity condition:

$$\int_0^{2\pi}\frac{\partial V_0}{\partial\theta}d\theta=\int_0^{2\pi}\left[\frac{1}{1-\nu^2}\left[\frac{1}{k^2}\frac{\partial^2\phi_0}{\partial s^2}-\nu\frac{\partial^2\phi_0}{\partial\theta^2}\right]+W_0\right]d\theta=0 \tag{16}$$

order of ϵ:

$$r^2\left[\frac{1}{k^4}\frac{\partial^4 W_1}{\partial s^4}+\frac{2}{k^2}\frac{\partial^4 W_1}{\partial s^2\partial\theta^2}+\frac{\partial^4 W_1}{\partial\theta^4}\right]+\lambda_0\frac{\partial^2 W_1}{\partial\tau^2}-\frac{1}{k^2}\frac{\partial^2 v_1}{\partial s^2}=-\lambda_1\frac{\partial^2 W_0}{\partial\tau^2}$$
$$+\frac{1}{k^2}\left[\frac{\partial^2\phi_0}{\partial\theta^2}\frac{\partial^2 W_0}{\partial s^2}-2\frac{\partial^2\phi_0}{\partial s\partial\theta}\frac{\partial^2 W_0}{\partial s\partial\theta}+\frac{\partial^2\phi_0}{\partial s^2}\frac{\partial^2 W_0}{\partial\theta^2}\right] \tag{17a}$$

$$\frac{1}{1-\nu^2}\left[\frac{1}{k^4}\frac{\partial^2\phi_1}{\partial s^4}+\frac{2}{k^4}\frac{\partial^4\phi_1}{\partial s^2\partial\theta^2}+\frac{\partial^4\phi_1}{\partial\theta^4}\right]+\frac{1}{k^2}\frac{\partial^2 W_1}{\partial s^2}=\frac{1}{k^2}\left[\left[\frac{\partial^2 W_0}{\partial\theta\partial s}\right]^2-\frac{\partial^2 W_0}{\partial s^2}\frac{\partial^2 W_0}{\partial\theta^2}\right] \tag{17b}$$

boundary conditions:

$$W_1=\frac{1}{k^2}\frac{\partial^2 W_1}{\partial s^2}+\nu\frac{\partial^2 W_1}{\partial\theta^2}=\frac{\partial^2\phi_1}{\partial\theta^2}=V_1=0 \quad \text{at } s=0,1 \tag{18}$$

circumferential periodicity condition:

$$\int_0^{2\pi}\frac{\partial V_1}{\partial\theta}d\theta=\int_0^{2\pi}\left[\frac{1}{1-\nu^2}\left[\frac{1}{k^2}\frac{\partial\phi_1}{\partial s^2}-\nu\frac{\partial^2\phi_1}{\partial\theta^2}\right]+\left[W_1-\tfrac{1}{2}\left[\frac{\partial W_0}{\partial\theta}\right]^2\right]\right]d\theta=0 \tag{19}$$

order of ϵ^2:

$$r^2\left[\frac{1}{k^4}\frac{\partial^2 W_2}{\partial s^4}+\frac{2}{k^2}\frac{\partial^4 W}{\partial s^2\partial\theta^2}+\frac{\partial^4 W}{\partial\theta^4}\right]+\lambda_0\frac{\partial^2 W_2}{\partial\tau^2}-\frac{1}{k^2}\frac{\partial^2\phi_2}{\partial s^2}=\frac{\bar{f}}{\epsilon^2}\cos\tau\cos(n\theta)\sin(m\pi s)$$
$$-\lambda_1\frac{\partial^2 W_1}{\partial\tau^2}-\lambda_2\frac{\partial^2 W_0}{\partial\tau^2}-\frac{\gamma}{\epsilon^2}\sqrt{\lambda_0}\sqrt{\lambda}\frac{\partial W_0}{\partial\tau}+\frac{1}{k^2}\left[\frac{\partial^2\phi_0}{\partial\theta^2}\frac{\partial^2 W_1}{\partial s^2}+\frac{\partial^2\phi_1}{\partial\theta^2}\frac{\partial^2 W_0}{\partial s^2}\right.$$
$$\left.-2\frac{\partial^2\phi_0}{\partial\theta\partial s}\frac{\partial^2 W_1}{\partial\theta\partial s}-2\frac{\partial^2\phi_1}{\partial\theta\partial s}\frac{\partial^2 W_0}{\partial\theta\partial s}+\frac{\partial^2\phi_0}{\partial s^2}\frac{\partial^2 W_1}{\partial\theta^2}+\frac{\partial^2\phi_1}{\partial s^2}\frac{\partial^2 W_0}{\partial\theta^2}\right] \tag{20a}$$

$$\frac{1}{1-\nu^2}\left[\frac{1}{k^4}\frac{\partial^4\phi_2}{\partial s^4}+\frac{2}{k^2}\frac{\partial^4\phi_2}{\partial s^2\partial\theta^2}+\frac{\partial^4\phi_2}{\partial\theta^4}\right]+\frac{1}{k^2}\frac{\partial^2 W_2}{\partial s^2}=\frac{1}{k^2}\left[2\frac{\partial^2 W_0}{\partial\theta\partial s}\frac{\partial^2 W_1}{\partial\theta\partial s}-\frac{\partial^2 W_0}{\partial s^2}\frac{\partial^2 W_1}{\partial\theta^2}-\frac{\partial^2 W_0}{\partial\theta^2}\frac{\partial^2 W_1}{\partial s^2}\right] \tag{20b}$$

boundary conditions:

$$W_2 = \frac{1}{k^2}\frac{\partial^2 W_2}{\partial s^2} + \nu\frac{\partial^2 W_2}{\partial \theta^2} = \frac{\partial^2 \phi_2}{\partial \theta^2}$$

$$= V_2 = 0 \quad \text{at } s=0,\ 1 \tag{21}$$

circumferential periodicity condition:

$$\int_0^{2\pi}\frac{\partial V_2}{\partial \theta}\,d\theta = \int_0^{2\pi}\Bigg[\frac{1}{1-\nu^2}\left[\frac{1}{k^2}\frac{\partial^2\phi_2}{\partial s^2} - \nu\frac{\partial^2\phi_2}{\partial\theta^2}\right]$$

$$+\left[W_2 - \frac{\partial W_0}{\partial\theta}\frac{\partial W_1}{\partial\theta}\right]\Bigg]d\theta = 0 \tag{22}$$

Similarly the linearized equations for the order of ϵ^3 and higher can be obtained in the same fashion.

III. Solutions of the Linearized Equations

The zero-order equations, Eqs. (14), are identical to the linear equations of free vibration of cylindrical shells. The solution can be readily obtained as

$$W_0 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}[A_{mn}(\tau)\cos(n\theta)$$

$$+B_{mn}(\tau)\sin(n\theta)]\sin(m\pi s) \tag{23}$$

If the shell is vibrated near a resonance frequency and the damping is small, the response will be dominated by the resonant mode for any reasonable spatial distribution of the forcing function. This case will be considered. Therefore,

$$W_0(s,\theta,\tau) = [A(\tau)\cos(n\theta) + B(\tau)\sin n\theta]\sin(m\pi s) \tag{24}$$

where

$$A(\tau) = a\cos(\tau+\delta_a) \quad B(\tau) = b\sin(\tau+\delta_b) \tag{25}$$

and δ_a and δ_b are the phase angles which are included in the response due to the damping and nonlinear effects. Here $A(\tau)\cos(n\theta)\sin(m\pi s)$ will be referred to as the driven mode since it is spatially similar to the external forcing function. $B(\tau)\sin(n\theta)\sin(m\pi s)$ will be referred to as the companion mode since it is excited indirectly.

The zero-order stress function ϕ_0 and nondimensional frequency λ_0 can be obtained by simple substitution of Eq. (24) into Eq. (14) as follows

$$\phi_0(s,\theta,\tau) = \Gamma[A(\tau)\cos(n\theta) + B(\tau)\sin(n\theta)]\sin(m\pi s) \tag{26}$$

$$\lambda_0 = r^2[(m\pi/k)^2+n^2]^2 + \frac{(1-\nu^2)(m\pi/k)^4}{[(m\pi/k)^2+n^2]^2} \tag{27}$$

where

$$\Gamma = [(1-\nu^2)\zeta^2/n^2(1+\zeta^2)^2], \quad \zeta = (m\pi/nk) \tag{28}$$

The amplitudes and phase angles of the driven mode and companion mode are not determined by this zero-order solution. Subsequent solutions of higher order equations will determine these values.

With the results of zero-order solution, the first-order equations, Eq. (17), together with boundary condition, Eq. (18), and circumferential periodicity condition, Eq. (19) can be solved. The solution may be expressed as follows

$$W_1(s,\theta,\tau) = \alpha_1(\tau)\cos(2m\pi s) + \alpha_2(\tau)\cos(2n\theta)$$
$$+\alpha_3(\tau)\sin(2n\theta) + g_0(\tau) + g_1(s) + g_2(s)[a^2\cos 2(\tau+\delta_a)$$
$$-b^2\cos 2(\tau+\delta_b)] + g_3(s)\cos(2n\theta) + g_4(s)\cos(2n\theta)$$
$$\times[a^2\cos 2(\tau+\delta_a) + b^2\cos 2(\tau+\delta_b)] + g_5(s)$$
$$\times\sin(2n\theta)\sin(2\tau+\delta_a+\delta_b)$$
$$+g_6(s)\sin(2n\theta)\sin(\delta_b-\delta_a)$$
$$\phi_1(s,\theta,\tau) = \bar\alpha_1(\tau)\cos(2m\pi s) + \bar\alpha_2(\tau)\cos(2n\theta)$$
$$+\bar\alpha_3(\tau)\sin(2n\theta) + \bar g_0(\tau) + \bar g_1(s) + \bar g_2(s)[a^2\cos 2(\tau+\delta_a)$$
$$-b^2\cos 2(\tau+\delta_b)] + \bar g_3(s)\cos(2n\theta) + \bar g_4(s)\cos(2n\theta)$$
$$\times[a^2\cos 2(\tau+\delta_a) + b^2\cos 2(\tau+\delta_b)] + \bar g_5(s)$$
$$\times\sin(2n\theta)\sin(2\tau+\delta_a+\delta_b)$$
$$+\bar g_6(s)\sin(2n\theta)\sin(\delta_b-\delta_a) \tag{29}$$

The functions α_i and $\bar\alpha_i$ where $i=1$, 2, 3, and g_i and $\bar g_i$ where $i=1,2,\ldots 6$ have lengthy expressions. They will not be included in the present report but can be found in Ref. 18, together with the detailed solution procedure.

During the solution procedure of the first-order equation, a secular term in the solution is encountered which grows indefinitely as $\tau\rightarrow\infty$. To eliminate this secular term, it is necessary to assign

$$\lambda_1 \equiv 0 \tag{30}$$

The solutions for the first-order perturbation equations as expressed by Eq. (29) include the particular solutions and the homogeneous solutions of the differential equations, Eq. (17). These solutions satisfy the specified boundary conditions, Eq. (18), as well as the circumferential periodicity condition, Eq. (19). Combining the zero and first-order solutions, the response of the large amplitude vibration of a cylindrical shell can be obtained to the accuracy of the order of $\epsilon=(w_m/R)$. This solution up to the order of ϵ is consistent with the approximate shell theory used in the analysis. However the response-frequency relationship has yet to be determined. The second-order perturbation equation will be used to obtain this relationship.

IV. Response-Frequency Relationship from Second-Order Equations

Since only the response-frequency relationship is sought from the second-order equation, so-called "secular terms" are of particular interest; in other words, the terms with spatial dependents $\cos(n\theta)\sin(m\pi s)$ and $\sin(n\theta)\sin(m\pi s)$. To collect all the secular terms, the first-order solutions have to be expanded into a Fourier series with variable s. A detailed solution procedure can be found in Ref. 18. To eliminate the secular terms, the following conditions must be satisfied.

$$(\Omega-1) + \left(\frac{h}{k}\right)^2\frac{\alpha}{s_0}w_a^2 + \frac{[\beta+\sigma\cos\Delta]}{\lambda_0}$$

$$\times\left(\frac{h}{k}\right)^2 w_b^2 + \frac{G\cos\delta_a}{\lambda_0 w_a} = 0 \tag{31}$$

$$\Omega\gamma + \left(\frac{h}{k}\right)^2\frac{\eta}{\lambda_0}w_b^2\sin\Delta + \frac{G\sin\delta_a}{\lambda_0 w_a} = 0 \tag{32}$$

$$w_b\Bigg[(\Omega^2-1) + \left(\frac{h}{k}\right)^2\frac{\alpha}{\lambda_0}w_b^2$$

$$+\frac{[\beta+\sigma\cos\Delta]}{\lambda_0}\left(\frac{h}{k}\right)^2 w_a^2\Bigg] = 0 \tag{33}$$

$$w_b\left[\Omega\gamma-\left(\frac{h}{k}\right)^2\frac{\eta}{\lambda_0}w_a^2\sin\Delta\right]=0 \tag{34}$$

where w_a and w_b are the normalized amplitudes of the driven mode and companion mode respectively. They are defined as

$$w_a=(w_m\cdot a/h),\quad w_b=(w_m\cdot b/h) \tag{35}$$

where a and b are defined in Eq. (25). Also

$$\Omega=\frac{\omega}{\omega_n},\ \Omega^2=\frac{\lambda}{\lambda_0}=1+\epsilon^2\frac{\lambda_2}{\lambda_0} \tag{36}$$

$$G=\frac{(1-\nu^2)}{E}\frac{k^2}{h^2}f,\ \Delta=2(\delta_b-\delta_a) \tag{37}$$

Equations (31-34) are 4 algebraic equations for four unknowns, namely the amplitudes w_a and w_b and the corresponding phase angles δ_a and δ_b. These unknowns can be solved in terms of α, β, σ, η, which are functions of circumferential wave number n, half axial wave number m, Poisson's ratio ν, length to radius ratio k, and nondimensional bending rigidity r. Their expressions can be found in Ref. 18. Also the solutions of the 4 equations involve the amplitude of external forcing function f and the damping γ. However the most important point indicated by these equations is that the amplitudes w_a and w_b, and their corresponding phase angle δ_a and δ_b are functions of the frequency Ω. Therefore, these are the resulting equations which provide the response-frequency relationship.

V. Driven Mode Response

One of the solutions of Eqs. (33) and (34) is $w_b\equiv 0$. In this case the companion mode is not participating in the vibration. The response involves the single mode, namely the driven mode only. Since Eqs. (33) and (34) are identically satisfied, Eqs. (31) and (32) can be reduced to a single equation by eliminating δ_a as follows

$$[(\Omega^2-1)+\left(\frac{h}{k}\right)^2\frac{\alpha}{\lambda_0}w_a^2]^2+\gamma^2\Omega^2=\frac{G^2}{\lambda_0^2w_a^2} \tag{38}$$

Equation (38) is the so-called response-frequency relationship, and Fig. 2 is a typical plot for a certain shell configuration and wave numbers. As can be seen, the nonlinearity is of the softening type, i.e., the frequency decreases with increasing response amplitude. This curve is similar to the response-frequency relationship of a simple spring-mass system and a nonlinear softening type spring as described in Ref. 1. Physically, when the frequency is increasing the response will follow the curve *a-b-c* and then jump to *d*. Further increasing the frequency, the response will follow the curve from *d* to *f*. When the frequency is decreasing the response will follow the curve *f-d-e* and then jump to *b*. Further decreasing the frequency, the response will follow the curve from *b* to *a*. The discontinuous responses at *c, d,* and *e, b* are called "jump phenomena" which have been observed in various nonlinear systems.

In the case of free vibration, the response-frequency relationship can be obtained by letting $G=\gamma=0$ in Eq. (38), namely

$$\Omega^2=1-(h/k)^2(\alpha/\lambda_0)w_a^2 \tag{39}$$

which is plotted as a dashed line in Fig. 2 and referred to as the "backbone curve." The backbone curve can be used to determine the type of nonlinearities (softening or hardening) and the degree of nonlinearity.

Although a softening type nonlinearity is indicated in Fig. 2, not all the modes (combination of m and n) are of softening type. An examination of Eq. (39) shows that the type of nonlinearity depends on the sign of the quantity α. Positive α implies a softening type nonlinearity and negative α implies a hardening type nonlinearity. Also the degree of nonlinearity is dependent on the magnitude of α.

VI. Companion Mode Participation

In the case of the companion mode participating in the vibration, i.e., $w_b\neq 0$, Eqs. (31-34) are solved numerically. Figure 3 shows the amplitude of the driven mode w_a with and without the companion mode participation for constant external force G and damping γ. Since $w_b\equiv 0$ is a solution of Eqs. (31-34), both curves in the figure represent the possible response of the shell. The actual response in the multi-valued region must be determined by a stability analysis. Figure 4 shows the corresponding amplitude of the companion mode as a function of Ω for constant G and γ. As can be observed, the response of the driven mode is affected by the companion mode only in a narrow region in the vicinity of $\Omega=1$. Its amplitude w_a is always greater than a certain value in this region. This minimum value can be derived as the minimum w_a necessary to make w_b a real value. From Eqs. (31-34) it is possible to obtain this minimum value as

$$w_a^2\geqslant(\lambda_0R^2/\eta)\Omega\gamma \tag{40}$$

This indicates that larger damping γ will make the companion mode occur only at larger amplitude.

After establishing the minimum amplitude for the driven mode, a minimum magnitude of forcing function can also be established. The existence of a minimum forcing function implies that to make the companion mode participate, the external forcing function must be greater than a certain minimum value. This explains the fact that the companion mode can be observed experimentally only for the large external excitation. It may be concluded that the participation

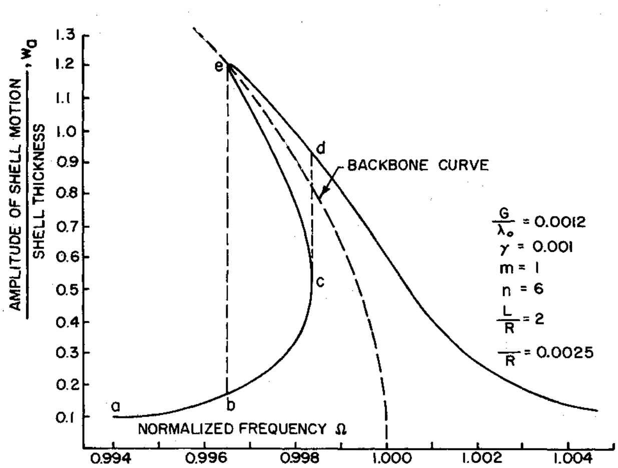


Fig. 2 Response-frequency relationship for driven mode only.

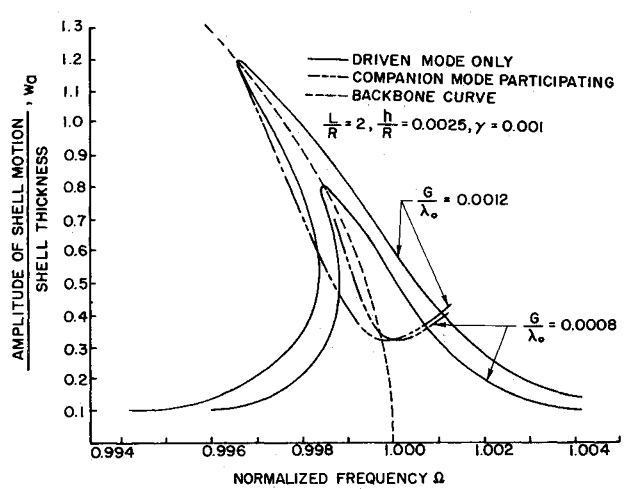


Fig. 3 Theoretical response-frequency relationship of driven mode $m=1$, $n=6$.

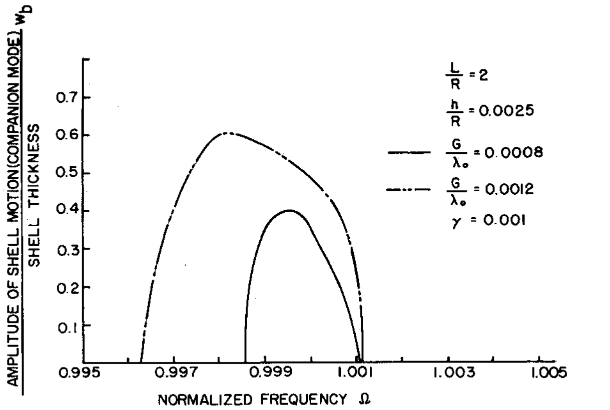


Fig. 4 Theoretical response-frequency relationship of companion mode $m=1$, $n=6$.

of the companion mode is a phenomenon of large amplitude and large external excitation.

VII. Mode Shape

The radial deflection for nonlinear vibration of a cylindrical shell simply supported at both ends up to order ϵ can be written as

$$W(s,\theta,\tau) = [a\cos(\tau+\delta_a)\cos n\theta + b\sin \sin n\theta]\sin \pi s$$
$$+\epsilon\{g_0(\tau)+\alpha_1(\tau)\cos 2m\pi s+\alpha_2(\tau)\cos 2n\theta+\alpha_3(\tau)\sin 2n\theta$$
$$+g_1(s)+g_2(s)[a^2\cos 2(\tau+\delta_a)-b^2\cos 2(\tau+\delta_b)]$$
$$+g_3(s)\cos 2n\theta+g_4(s)\cos 2n\theta[a^2\cos 2(\tau+\delta_a)+b^2\cos 2(\tau+\delta_b)]$$
$$+g_5(s)\sin 2n\theta\sin(2\tau+\delta_a+\delta_b)+g_6(s)\sin 2n\theta\sin(\delta_b-\delta_a)\}$$
$$+O(\epsilon^2)+\ldots \tag{41}$$

where the amplitude of the driven mode a and the companion mode b are functions of the frequency, external forcing function and damping. The terms multiplied by the small parameter ϵ are due to the nonlinearities of the problem. These terms represent the difference of the deflection shape and harmonic content between the linear and nonlinear response of the shell. Therefore, it is of interest to discuss these terms. The functions $g_0(\tau)$, $\alpha_1(\tau)$, $\alpha_2(\tau)$ and $\alpha_3(\tau)$ involve a constant term and a term with $\cos 2\tau$ or $\sin 2\tau$. The functions $g_1(s)$ to $g_6(s)$ are of the "boundary-layer" type, i.e., they possess large magnitude near both ends ($s=0$ and $s=1$) and vanishingly small magnitude elsewhere. Therefore, as far as the deflection shape away from the immediate vicinity of the boundary is concerned, terms like $g_0(\tau)$, $\alpha_1(\tau)\cos 2m\pi s$, $\alpha_2(\tau)\cos 2n\theta$ and $\alpha_3(\tau)\sin 2n\theta$ are dominant. A detailed comparison of the response form between the present investigation and previous investigations can be found in Ref. 18.

The presence of the companion mode is an interesting phenomenon. The appearance of traveling wave response due to the companion mode can be explained as follows. Let

$$W(\theta,\tau) = a\cos\tau\cos(n\theta) + b\sin\tau\sin(n\theta) \tag{42}$$

The axial dependence is omitted for convenience. This can be rearranged as

$$W(y,t) = b\cos\left(\frac{ny}{R} - \omega t\right) + c\cos\omega t\cos\frac{ny}{R} \tag{43}$$

where $c=a-b$. Defining the circumferential wave length as

$$\lambda_c = \frac{2\pi R}{n}$$

then

$$W(y,t) = b\cos\frac{2\pi}{\lambda_c}\left(y - \frac{R\omega}{n}t\right) + c\cos\omega t\cos\frac{ny}{R} \tag{44}$$

This is a traveling wave form with wave length λ_c and phase velocity $R\omega/n$ in./sec. Now the period T required for this wave to travel a complete circumference (1 cycle) is

$$T = 2\pi R/(R\omega/n) = 2n\pi/\omega \text{ sec} \tag{45}$$

hence the frequency of the traveling wave ω_T

$$\omega_T = \omega/2n\pi \text{ cps} = \omega/n \text{ rad/sec} \tag{46}$$

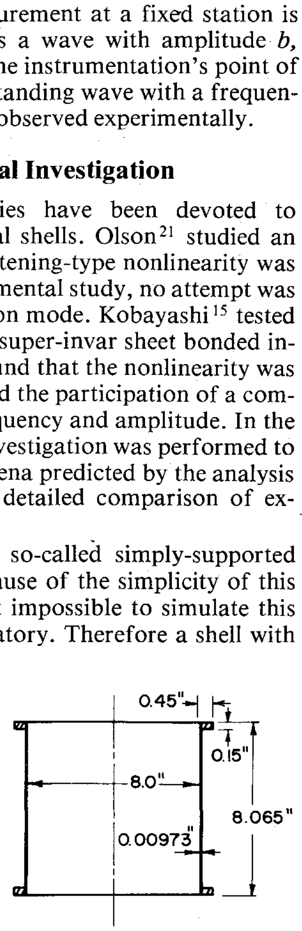
Therefore, a displacement measurement at a fixed station is the standing wave $c\cos(\omega t)$ plus a wave with amplitude b, coming every $2n\pi/\omega$ sec. From the instrumentation's point of view this appears to be another standing wave with a frequency ω/n. The phenomena is easily observed experimentally.

VIII. Experimental Investigation

Very few experimental studies have been devoted to nonlinear vibration of cylindrical shells. Olson[21] studied an electroplated shell in which a softening-type nonlinearity was observed. However, in his experimental study, no attempt was made to investigate the companion mode. Kobayashi[15] tested a shell which was constructed of super-invar sheet bonded into a seamed cylinder. He also found that the nonlinearity was of the softening type and observed the participation of a companion mode over a range of frequency and amplitude. In the present study, an experimental investigation was performed to observe qualitatively the phenomena predicted by the analysis and wherever possible to make detailed comparison of experiment and theory.

In the theoretical analysis, a so-called simply-supported shell (SS1/SS1) was studied because of the simplicity of this boundary condition. It is almost impossible to simulate this boundary condition in the laboratory. Therefore a shell with

Fig. 5 Properties of shell-ring specimen. Density $\rho=0.101$ 16/in.^{3}, Young's modulus $E=10.3\times10^6$ psi, and Poisson's ratio $\nu=0.31$.



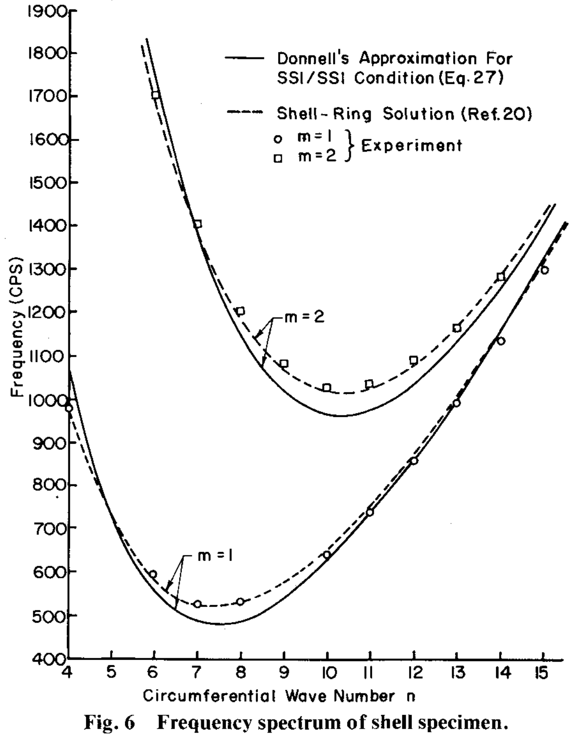


Fig. 6 Frequency spectrum of shell specimen.

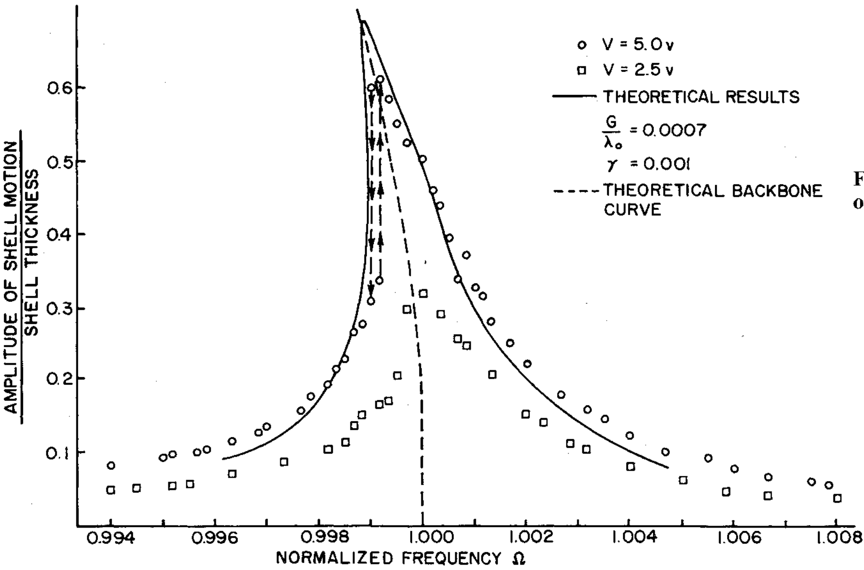


Fig. 7 Experimental response-frequency relationship of driven mode $m=1$, $n=6$, of small external force.

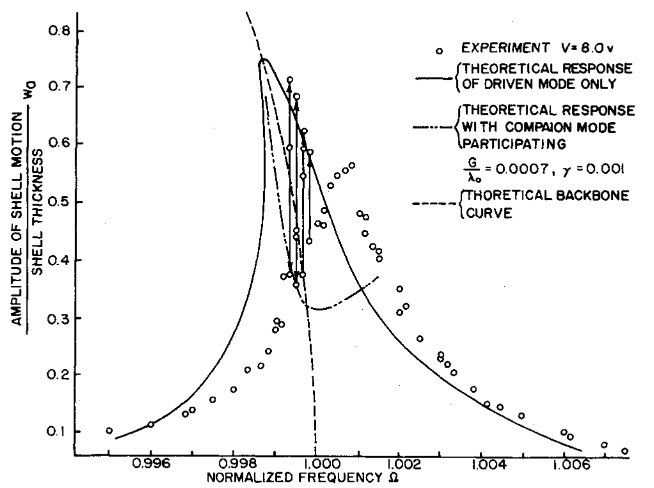


Fig. 8 Experimental response-frequency relationship of driven mode $m=1$, $n=6$, of higher external force.

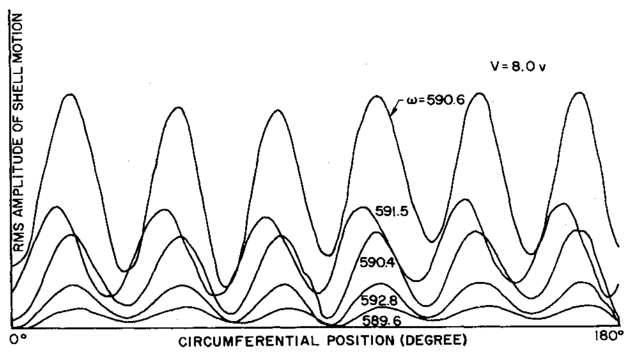


Fig. 9 Mode shapes for $m=1$, $n=6$.

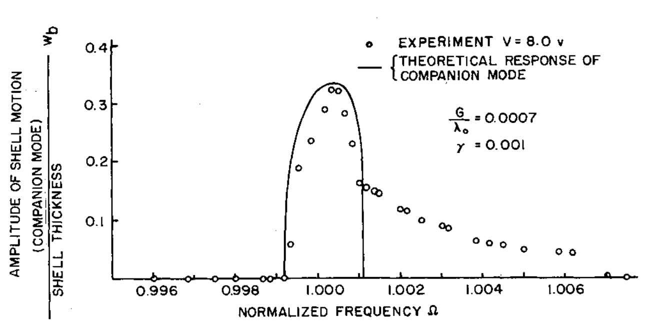


Fig. 10 Experimental response-frequency relationship of companion mode $m=1$, $n=6$.

rings at both ends was chosen to approximate the simply-supported condition. The shell-ring system is shown in Fig. 5. The mode shapes and frequencies of natural vibration of this shell-ring system[20] was analyzed and compared with that of a corresponding simply-supported shell. The comparison indicated that the shell-ring system is indeed an approximation of a simply-supported shell. Figure 6 shows the comparison of frequencies together with experimentally obtained frequencies of the test specimen. Vibrations of the shell-ring system are excited by an acoustic driver and the motions of the shell skin is measured with a noncontact reluctance-type pickup.

The $m=1$ modes were used for response-frequency relationship study because these modes are the easiest to excite into a nonlinear region. The magnitude of the force output of the acoustic driver was kept relatively constant over the narrow frequency range of interest by applying a constant voltage across the driver. At each data point the frequency measured by an electronic counter was normalized by the linear frequency ω_0 which is known prior to the test, and the response measurement was converted into physical value (i.e., displacement) and then normalized by the shell thickness. Figure 7 shows the response-frequency relation for $m=1$, $n=6$ mode. When the external force is "small," no jump phenomena is detected. But the "jump" is detected when the external force is increased. Corresponding theoretical results are also plotted which show satisfactory agreement with the experimental results. With the magnitude of the acoustic input increased further, a different response-frequency relationship was obtained as shown in Fig. 8. This result indicates that in the vicinity of the resonance frequency the response became "nonstationary" in the sense that the response would not remain at one amplitude, rather it drifted slowly from one amplitude to another. In some cases as many as 4 different amplitudes were observable at a given frequency. It was obvious that at these frequencies the driven mode response became multi-valued. For this particular test, the region of multi-response was at $\Omega<1$. A circumferential mode plotted at constant external force for different frequencies is shown in Fig. 9. As the mode shape indicates, for larger amplitude the response was no longer a single driven mode $\cos(n\theta)$. At the node point of $\cos(n\theta)$, which is the antinode of $\sin(n\theta)$, the deflection did not vanish. This was the first indication that the companion mode was participating in the vibration.

The response-frequency relation of the companion mode was also measured and is shown in Fig. 10. The experimental results compared with the theoretical results show some discrepancies for $\Omega<1$. This may be because of the following facts. The amplitude of the companion mode was measured at a predetermined node of the driven mode $\cos(n\theta)$. However, the shifting of the node as shown in Fig. 9 made the com-

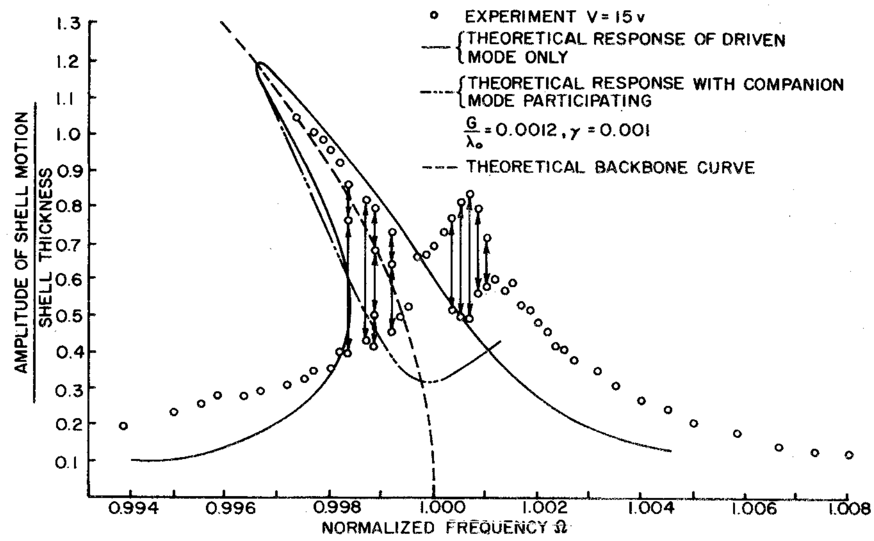


Fig. 11 Nonstationary response-frequency relationship of driven mode $m=1, n=6$.

panion mode measurement "contaminated" by some amount of the amplitude of the driven mode.

The participation of the companion mode has changed the characteristics of nonlinear vibration of the cylindrical shell as compared to the case of driven mode response only. The "jump phenomenon" has been eliminated and instead a "nonstationary" response is observed in which the amplitude drifts from one value to another.

An attempt was made to constrain the companion mode by lightly placing a sharp point at the antinode of the companion mode $\sin(n\theta)$ and then taking measurements at the antinode of the driven mode $\cos(n\theta)$. The classical "jump phenomena" was recovered and the shell vibrated in the driven mode only.

With the voltage of the acoustic driver further increased, another test was conducted, and the response-frequency relation was obtained as shown in Fig. 11. Again there was a narrow region in the vicinity of $\Omega=1$ in which the response drifted between several values. Within the narrow frequency region where the drifting response was found, a few points very close to $\Omega=1$ seemed to be "stationary" or the drifting rate was too small to detect. The high-amplitude response found outside of the drifting region ($\Omega<1$) was extremely difficult to obtain, since the amplitude had a tendency to drop to the lower value.

In these experiments no attempt was made to measure the force exerted by the acoustical driver. Therefore, in making the comparisons with the theory as shown in Figs. 7, 8, 10, and 11, the force parameter G was selected in order to obtain the best comparison with the experiment. The damping parameter γ was likewise selected. By a proper choice of these parameters, qualitative agreement with the experiment can be demonstrated. This comparison is most favorable with the lower amplitude at the "nonstationary" response. This type of response was not studies theoretically.

IX. Conclusions

The theoretical treatment of this problem by the perturbation method has yielded qualitatively most of the nonlinear phenomena demonstrated experimentally. The advantage of this method over previous solution techniques, which demand an initial assumption on the vibration mode, is that the results are not pre-biased. The resulting single mode response is either hardening or softening depending upon the mode of vibration (m, n). The companion mode phenomena, as pointed out by previous investigators, is recovered. However, additional insight is gained into the nonlinear coupling that occurs and the mode of response at the frequency 2ω. The "nonstationary response" detected experimentally may either be fundamental to the problem or a result of the particular experiment. Experimental work of this type appears to be very sensitive to nonideal conditions whether geometrical of thickness deviation of the shell, support conditions or the forcing function. Further refinement of these experimental "errors" or inclusion in theoretical analysis would perhaps enhance the quantitative agreement, but is is doubted if the fundamental behavior of the driven mode-companion mode interaction would be altered.

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